Math 53, Discussion 105

Monday, April 27, 2020 9:33 AM

16.8 (Stokes' theorem)
16.9 (Divergence theorem)

$$d\vec{S} = \vec{n}\, dS = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \|\vec{r}_u \times \vec{r}_v\|\, dA = \vec{r}_u \times \vec{r}_v\, dA$$

(1 point) **local/Library/Michigan/Chap19Sec3/Q07.pg**

Compute the flux of the vector field $\vec{F} = 3x^2y^2z\vec{k}$ through the surface S which is the cone $\sqrt{x^2+y^2} = z$, with $0 \le z \le R$, oriented downward.

(a) Parameterize the cone using cylindrical coordinates (write θ as *theta*).

$x(r,\theta) =$

$y(r,\theta) =$

$z(r,\theta) =$

with $\le r \le$

and $\le \theta \le$

(b) With this parameterization, what is $d\mathbf{S} = d\vec{S} = \vec{n} \cdot dS$)? Your answer should involve dr and dtheta (with no space between the d and the variable).

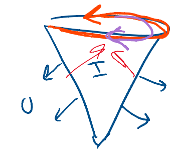
$d\vec{S}$ =

want $d\vec{S}$ with negative z component

(c) Find the flux of $\vec{F}$ through S.

flux =

Solution:



$r\cos\theta,\ r\sin\theta,\ r$

Theorem (Stokes)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

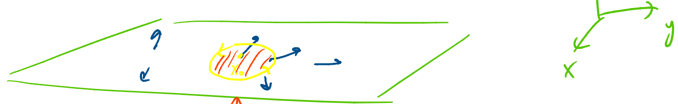
note: C positively oriented & closed

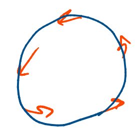
S is any surface with boundary C

Interpret to give a meaning of curl

Given vector field in $\mathbb{R}^3$

To find curl at x_0 in z direction integrate a small loop around x_0 in the x-y plane





A = surface

$\Rightarrow d\vec{S} = (0,0,1)\, dS$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \iint_S \underbrace{(\nabla \times F) \cdot (0,0,1)}\, dS = (\text{Area of } A) \cdot \text{Avg of curl in } z \text{ direction}$$

z compon of $\nabla \times F$

Theorem (Divergence theorem)

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{Surface integral}} = \overbrace{\iiint_V \nabla \cdot \vec{F}\, dV}^{\text{volume integral}}$$

divergence of F

volume enclosed by S

Any closed surface

$$\boxed{\begin{array}{l}\text{Note:} \\ \nabla \cdot F(x_0) > 0 \Rightarrow \text{Source at } x_0 \\ \nabla \cdot F(x_0) < 0 \Rightarrow \text{sink at } x_0\end{array}}$$

V = Cube $\quad \nabla \cdot F = C$

$\text{Vol}(V) = 1$

$\Rightarrow$ RHS $= C$

LHS $= C$

LHS = flux of F through S

RHS = (Avg divergence of F)(volume of V)

(1 point) **Library/Michigan/Chap20Sec2/Q37.pg**

As a result of radioactive decay, heat is generated uniformly throughout the interior of the earth at a rate of around 30 watts per cubic kilometer. (A watt is a rate of heat production.) The heat then flows to the earth's surface where it is lost to space. Let $\vec{F}(x, y, z)$ denote the rate of flow of heat measured in watts per square kilometer. By definition, the flux of $\vec{F}$ across a surface is the quantity of heat flowing through the surface per unit of time.

(a) Suppose that the actual heat generation is $28\mathrm{W/km}^3$. What is the value of div $\vec{F}$?

div $\vec{F}$ =

(Include units.)

(b) Assume the heat flows outward symmetrically. Verify that $\vec{F} = \alpha\vec{r}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and α is a suitable constant, satisfies the given conditions. Find α.

$\alpha =$

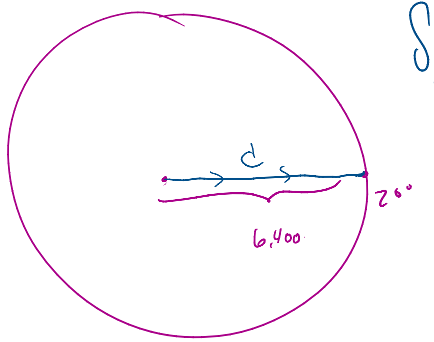
(Include units.)

(c) Let $T(x, y, z)$ denote the temperature inside the earth. Heat flows according to the equation $\vec{F} = -k$ grad T, where k is a constant. If T is in °C, then $k = 28{,}000$°C/km. Assuming the earth is a sphere with radius 6400 km and surface temperature 20°C, what is the temperature at the center?

$T =$ (degrees C)

$\vec{F} = -k\,\nabla T$

F



$\int_C F\cdot dr = -k\,(T(\text{surf}) - T(0))$

$= -k\,(20 - T(0))$

$r = (6{,}400\,t,\ 0,\ 0)$

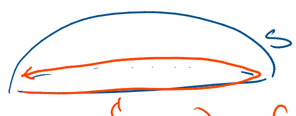
$\int_C F\cdot dr = \int P\,dx + Q\,dy + R\,dz = \int_0^1 \alpha(6{,}400\,t)\ 6{,}400\ dt$

$dy = 0$

$dz = 0$

$dx = 6{,}400$

$F = (\alpha x, \alpha y, \alpha z)$



Exercises:

1) Compute $\iint_S \nabla\times \boldsymbol{F}\cdot d\boldsymbol{S}$ **for** $F =$ $\langle 2y\cos z, e^x \sin z, xe^y\rangle$, on the upper half of the sphere centered at the origin with radius 9

2) Compute $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ for $F = \langle yz, 2xz, e^{xy}\rangle$ C the circle $x^2 + y^2 = 16, z = 5$

1) Compute $\iint_S F\cdot dS$ for $F = \langle exy^2, xe^z, z^3\rangle$ over the surface bounded by $y^2 + z^2 = 1$ and the $x = -1, x = 2$

(a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.

(b) Given that $\mathbf{F}(x, y) = \langle x, y^2\rangle$, use the definition of divergence to verify your answer to part (a).

1) $C = (9\cos\theta,\ 9\sin\theta,\ 0)$

$dz = 0$

$dy = 9\cos\theta$

$Q = 0$

$dx = -9\sin\theta$

$\oint_C F\cdot dr = \int_0^{2\pi} \boxed{P\,dx + \cancel{Q\,dy} + \cancel{R\,dz}}$

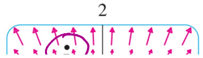
0

$= \int_0^{2\pi} 2(9\sin\theta)\ \cos(0)\ (-9\sin\theta)\ d\theta$

z y $\cos(z)$

$9\sin\theta$ 0

2

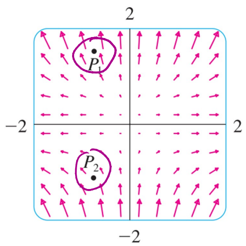


$\nabla\cdot F(P_1) > 0$

2)

gence to verify your answer to part (a).

2)



$\nabla \cdot F(P_1) > 0$

$\nabla \cdot F(P_2) < 0$